Exercise 1: Bias and variance of ill-posed inverse problems

Create a function to compute the $N \times N$ Hilbert-matrix

$$A_{ij} = \frac{1}{i+j-1}$$

given the dimension $N$ as input.

a) Generate simulated data $\vec{b}$ for normal distributed noise:

- choose:
  $$x_j = \sin(2\pi(j-1)/(N-1)), \quad j = 1, \ldots, N$$

- compute:
  $$\tilde{b}_i = \sum_{j=1}^{N} A_{ij} x_j$$

- add noise $\varepsilon_i \sim N(0, \sigma^2)$:
  $$b_i = \tilde{b}_i + \varepsilon_i$$

b) Estimate $\vec{x}$ from the data $\vec{b}$:

- Calculate the singular-value decomposition of $A$ (numpy.linalg.svd). How do the singular values relate to the eigenvalues in case of a symmetric matrix?

- Compute the condition number $\kappa(A)$ via numpy.amin and numpy.amax.

- Write a function to estimate $x_i$ for the given $b_i$ from Eq. (1), using the inverted matrix $A^{-1}$ based on its singular value decomposition:

  $$A^{-1} = V^T \text{diag}(1/w_i) U^T.$$ (2)

Use the built-in functions numpy.diag, numpy.divide, numpy.matmul and numpy.transpose.

c) Test several setups regarding the effect of the regularization, $1/w_i = 0$ if $\max(w)/w_i > \kappa_{\max}$, for different choices of the maximal condition number $\kappa_{\max}$, of $N$ and of the noise $\sigma$.

- $N = 4, \sigma = 0.001$ for either no regularization or for $\kappa_{\max} = 1000$

- $N = 7, \sigma = 10^{-5}, \kappa_{\max} \in \{100, 10^5, 10^{10}\}$

- $N = 10, \sigma = 0, \kappa_{\max} = \inf$ ("inf") in python. Compare the result with the one you obtain from using an in-build matrix inversion function (numpy.linalg.inv) instead of Eq. (2). How are $A^{-1}$ and the $x_i$ affected?

- $N = 42, \sigma = 0, \kappa_{\max} \in \{100, 10^5, 10^{10}\}$