Statistics and Numerics

Lecture Prof. Dr. Jens Timmer

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Exercise Sheet Nr. 9

Exercise 1: Numerical integration of the Van-der-Pol oscillator

The Van-der-Pol oscillator is described by the following second order differential equation:

$$\ddot{x} = \mu(1 - x^2)\dot{x} - x \tag{1}$$

- Transform Eq. (1) into two differential equations of first order.
- Implement your own Runge-Kutta 4th order algorithm to integrate the Van-der-Pol oscillator, given $\mu = 5$. Hence, write a function `int_RK4(f,t0,y0,tend,dt)`, which takes the function f to integrate (which maps to another function that you have to write that evaluates the two differential equations), the initial time $t0$, the initial states $y0$, the end time $tend$ and the step size of the integrator, dt.
  Details for y0, tend and dt:
  - The period of this system is around 10 time units. With this information, choose a good time interval for integration.
  - Choose initial values for Eq. (1) from $\mathcal{N}(0,1)$ and plot the result in time- and phase-space.
  - Explain what you see!
  - Choose different integration step sizes $dt = [0.001, 0.01, 0.1]$, and compare the integration times and quality of the solution. For this purpose, take the same initial values!
- Now, take $\mu = 35$, and do the integration again with your `rk4()` for different step sizes. Can you see a difference?
- Perform the integration, $\mu = 35$, with the built-in RK45 integrator of python. It features an adaptive step size and can be started via scipy.integrate.RK45(f,t0,y0,tend).
  - Once the integrator is initialized, you need a *while* loop with manual `RK45.step()` calls until the `RK45.status` differs from *running*. In this loop, you have to store the current time and state of the equations. You might also have a look at the solution online!
  - From the internal steps of the $RK45$ integrator, calculate the step sizes and plot them below your result of RK45 to see where the integrator had to choose small step sizes. Does this make sense?
- Now, set $\mu = 1000$, which results in a stiff system. Do the integration with `RK45()` and compare it with a dedicated integrator for stiff systems, `scipy.integrate.BDF`. Compare the computation times. Therefore, include `default_timer` from the `timeit` package, and start and stop the timer once for `RK45` and once for the `BDF` integrator.

Additional challenge: Spectral analysis

- Compute the numerical Fourier transform $\tilde{x}(f)$ of your solution $x(t)$ of the Van-der-Pol oscillator, given $\mu = 5$. Use `numpy.fft.fft()`.
- Calculate the periodogram $S(f)$, defined by the squared absolute values of the Fourier transform:

$$S(f) \sim |\tilde{x}(f)|^2.$$

- Take different intervals of your signal $x(t)$ with varying number of peaks. Interpret the result, also given the non-linearity of the underlying differential equations (Eq. (1)).

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