The Astrophysical Journal, 734:97 (8pp), 2011 June 20 doi:10.1088/0004-637X/734/2/97

A UNIFIED APPROACH TO THE HELIOSEISMIC INVERSION PROBLEM OF THE SOLAR MERIDIONAL FLOW FROM GLOBAL OSCILLATIONS

A. Schad[1,2,3], J. Timmer[1,2,4,5,6], and M. Roth[3]
[1] Freiburg Center for Data Analysis and Modeling (FDM), 79104 Freiburg, Germany
[2] Institute for Physics, University of Freiburg, 79104 Freiburg, Germany
[3] Kiepenheuer-Institut für Sonnenphysik (KIS), 79104 Freiburg, Germany
[4] Freiburg Institute for Advanced Studies (FRIAS), 79104 Freiburg, Germany
[5] BIOSS Centre for Biological Signalling Studies, University of Freiburg, 79104 Freiburg, Germany
[6] Department of Clinical and Experimental Medicine, Linköping University, SE-581 83 Linköping, Sweden
Received 2010 December 23; accepted 2011 April 5; published 2011 June 1

ABSTRACT

Measurements from tracers and local helioseismology indicate the existence of a meridional flow in the Sun with strength in the order of 15 m s^{-1} near the solar surface. Different attempts were made to obtain information on the flow profile at depths up to 20 Mm below the solar surface. We propose a method using global helioseismic Doppler measurements with the prospect of inferring the meridional flow profile at greater depths. Our approach is based on the perturbation of the *p*-mode eigenfunctions of a solar model due to the presence of a flow. The distortion of the oscillation eigenfunctions is manifested in the mixing of *p*-modes, which may be measured from global solar oscillation time series. As a new helioseismic measurement quantity, we propose amplitude ratios between oscillations in the Fourier domain. We relate this quantity to the meridional flow and unify the concepts presented here for an inversion procedure to infer the meridional flow from global solar oscillations.

Key words: methods: analytical – Sun: helioseismology – Sun: interior – Sun: oscillations

1. INTRODUCTION

The meridional flow is described by an axisymmetric poloidal zonal flow on the Sun. It can be measured on the Sun by tracking surface features (Wöhl 2002) or by determining the contribution of the surface meridional flow to the shift of solar spectral lines (Hathaway 1996). Inside the Sun, local helioseismic investigations based on ring diagram analysis and time–distance analysis of high-degree *p*-modes have shown that near the solar surface a poleward flow on both hemispheres can be observed with a strength between 10 and 20 m s^{-1} (cf. Haber et al. 2002; Zhao & Kosovichev 2004; Zaatri et al. 2006; Gizon 2004, and references therein). However, these investigations are restricted to the outer few Mm of the Sun due to the low penetration depth of high-degree *p*-modes. To maintain the mass balance of the Sun a counterflow at some greater depth which has not been detected yet is required.

Theoretical considerations based on perturbation analysis applied to solar models for *p*-modes have shown that large-scale flows in the solar interior, like convection, giant cells, and differential rotation result in couplings between the eigenfunctions of *p*-modes (Ritzwoller & Lavely 1991; Lavely & Ritzwoller 1992; Roth & Stix 1999, 2003, 2008; Woodard 2000; Roth et al. 2002). These couplings are manifested in splittings and shifts of the mode eigenfrequencies and distortions of the eigenfunctions. For example, differential rotation leads to a coupling mainly between degenerate modes of the same multiplet and results in frequency splittings described by degenerate perturbation theory. Measurements of frequency splittings from global solar oscillations are efficiently used for the seismic probing of the differential rotation (e.g., Howe 2009 and references therein). As shown in Lavely & Ritzwoller (1992), a poloidal convective motion is not able to couple degenerate modes. In that case, mode coupling is found between pairs of modes of similar oscillation frequency but originating from different multiplets. The coupling strength, however, is much smaller compared to those caused by the differential rotation. Quasi-degenerate perturbation analysis has shown that theoretical frequency shifts due to a meridional flow are of second order in the flow amplitude (Roth & Stix 2008). Forward modeling of the effect of the meridional flow on the *p*-modes shows that the frequency shifts for low-degree *p*-modes with $l \leqslant 300$ are expected to have values in the order of some nHz up to a few μHz (Roth & Stix 2008; Chatterjee & Antia 2009).

In the following, we shall focus on the perturbations of the eigenfunctions due to a slowly streaming large-scale meridional flow. We present a theoretical concept for setting up procedures to infer the meridional flow from global helioseismic observations of low-degree *p*-modes. As a first step, the formalisms used to model a meridional flow inside the Sun are introduced, and the concepts and results of quasi-degenerate perturbation applied to solar global oscillations are outlined. Then in Section 3, we derive approximations for the perturbed eigenvectors and relate them to the flow using non-degenerate perturbation theory. In Section 4, the perturbed eigenvectors are related to the observable quantities of global oscillation time series and the expected order of magnitude of the perturbation in the observable quantities is estimated. Finally, we discuss the presented concept and its use for the inversion of a meridional flow from a low-degree global oscillation time series.

2. THE MERIDIONAL FLOW AND QUASI-DEGENERATE PERTURBATION THEORY

The influence of a large-scale flow on the eigenfrequencies and eigenfunctions of *p*-modes can be treated as a perturbation of a standard solar equilibrium model. The perturbation leads to a coupling between *p*-modes. For a meridional flow, these couplings can be investigated by quasi-degenerate perturbation theory as discussed in Lavely & Ritzwoller (1992) and Roth & Stix (2008). In the following, we shortly summarize the mathematical concepts and results presented therein that are used in the following sections.

2.1. The Equilibrium Model

We consider a non-rotating, non-magnetic, static, and spherically symmetric equilibrium model of the Sun as described by Lavely & Ritzwoller (1992) and Roth & Stix (1999). The model supplies eigenfunctions $\boldsymbol{\xi}_k^0$ and eigenfrequencies ω_k of adiabatic oscillations as a solution of the eigenvalue equation:

$$\mathcal{L}_0 \boldsymbol{\xi}_k^0 = -\rho_0\, \omega_k^2\, \boldsymbol{\xi}_k^0 \,, \tag{1}$$

where ρ_0 is the mass density. The operator $\mathcal{L}_0$ is derived from the Eulerian perturbations of the solar model due to a displacement $\boldsymbol{\xi}$ of a parcel of gas exerted by Eulerian variations of pressure P', gravitational acceleration $\boldsymbol{g}'$, and density ρ':

$$\mathcal{L}_0 \boldsymbol{\xi} = -\nabla P' + \rho_0 \boldsymbol{g}' + \rho' \boldsymbol{g}_0 \,. \tag{2}$$

The superscript 0 is used to highlight quantities from the equilibrium model. In spherical coordinates with longitude ϕ and colatitude θ, the eigenfunctions $\boldsymbol{\xi}_k^0$ can be represented in vector spherical harmonics $\boldsymbol{Y}_l^m = (Y_l^m, \nabla_h Y_l^m)$, where Y_l^m is a spherical harmonic,

$$Y_l^m(\theta, \phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta) e^{im\phi} \,, \tag{3}$$

of harmonic degree $l \geqslant 0$ and azimuthal order $|m| \leqslant l$, and ∇_h denotes the horizontal gradient in θ and ϕ. The function P_l^m is the associated Legendre polynomial. The eigenfunctions are then written as

$$\boldsymbol{\xi}_k^0(r, \theta, \phi) = \xi_{nl}^r(r) Y_l^m(\theta, \phi) \boldsymbol{e_r} + \xi_{nl}^h(r) \nabla_h Y_l^m(\theta, \phi) \,, \tag{4}$$

where $\boldsymbol{e_r}$ is the radial unit vector and ξ^r and ξ^h are real valued scalar radial eigenfunctions for the radial and horizontal component of $\boldsymbol{\xi}_k^0$. The eigenfunctions are characterized by a triple of integers, $k = (n, l, m)$, where n refers to the radial order of the scalar eigenfunctions. The eigenfunctions are orthogonal

$$\int \rho_0\, \overline{\boldsymbol{\xi}_{k'}^0} \cdot \boldsymbol{\xi}_k^0\, d^3\boldsymbol{r} = N_k \delta_{k'k} \,, \tag{5}$$

where the bar denotes complex conjugation. We additionally assume throughout this paper that the eigenfunctions are normalized such that

$$N_k = \int \rho_0 \left[\left(\xi_{nl}^r(r)\right)^2 + l(l+1)\left(\xi_{nl}^h(r)\right)^2 \right] r^2 dr = 1 \,. \tag{6}$$

Modes with equal radial order and harmonic degree form multiplets (n, l) of oscillations with different azimuthal order m but with identical frequencies, i.e., $\omega_{\mathrm{nlm}} = \omega_{nl}$.

The time dependency of each mode is given by an oscillation function $\alpha_k(t)$ for its amplitude conventionally described by a stochastically driven damped oscillator with central frequency ω_k and damping Γ_k. Thus the general solution of the equation of motion is of the form $\boldsymbol{x}_k(\boldsymbol{r}, t) = \alpha_k(t) \boldsymbol{\xi}_k^0(\boldsymbol{r})$. The amplitude and the damping rate are not determined by standard solar models for global oscillations.

2.2. Quasi-degenerate Perturbation Theory

The coupling of modes due to a meridional flow is restricted to modes of different multiplets (n, l) and (n', l') but the same azimuthal order m (Lavely & Ritzwoller 1992; Roth & Stix 2008). These modes normally have unequal eigenfrequencies and the coupling strength increases with decreasing difference of the eigenfrequencies. The similarity of eigenfrequencies necessary for the coupling of modes is expressed by the quasi-degeneracy condition:

$$\left|\omega_{\mathrm{ref}}^2 - \omega_k^2\right| < \epsilon \delta\omega^2 \,, \tag{7}$$

where ω_{ref} is an arbitrary chosen reference frequency next or equal to the frequency of a multiplet $j = (n, l)$ under consideration. The parameter ϵ is an auxiliary parameter for the order of the perturbation and $\delta\omega$ restricts the frequency range where mode coupling may be effective. Condition (7) determines a subset K of coupling modes.

The perturbation of the equation of motion, eigenfunctions, and eigenfrequencies is obtained from the expansion:

$$\mathcal{L}_0 \to \mathcal{L}_0 + \epsilon \mathcal{L}_1 \tag{8}$$

$$\boldsymbol{\xi}_j^0 \to \tilde{\boldsymbol{\xi}}_j^0 + \epsilon \tilde{\boldsymbol{\xi}}_j^1 \tag{9}$$

$$\omega_j^2 \to \omega_{\mathrm{ref}}^2 + \epsilon \left(\omega_1^2\right)_j \,, \tag{10}$$

with $(\omega_1^2)_j$ as the squared eigenfrequency perturbation in first order. In quasi-degenerate perturbation theory, the zeroth-order and first-order eigenfunction perturbations can be expressed in terms of the unperturbed eigenfunctions:

$$\tilde{\boldsymbol{\xi}}_j^0 = \sum_{k \in K} c_{jk} \boldsymbol{\xi}_k^0 \tag{11}$$

$$\tilde{\boldsymbol{\xi}}_j^1 = \sum_{k \in K^\perp} b_{jk} \boldsymbol{\xi}_k^0 \,. \tag{12}$$

The expansion coefficients c_{jk} and b_{jk} may be complex valued and the set $K^\perp$ is the complement of K.

We focus on the coefficients c_{jk} needed to determine $\tilde{\boldsymbol{\xi}}_j^0$. Neglecting the perturbation in the structural solar quantities, like density, it can be shown that the coefficients c_{jk} are the elements of an eigenvector that satisfy the eigenvalue problem:

$$\sum_{k \in K} c_{jk} Z_{k'k} = \sum_{k \in K} c_{jk} \left(\omega_1^2\right)_j \delta_{k'k} \quad \text{for} \quad k' \in K \,, \tag{13}$$

with

$$Z_{k'k} = \begin{cases} H_{k'k} - \left(\omega_{\mathrm{ref}}^2 - \omega_k^2\right) \delta_{k'k} & \text{for} \quad k', k \in K \\ 0 & \text{otherwise} \,, \end{cases} \tag{14}$$

and the general matrix element

$$H_{k'k} = -\int \overline{\boldsymbol{\xi}_{k'}^0} \cdot \mathcal{L}_1\left(\boldsymbol{\xi}_k^0\right) d^3\boldsymbol{r} \,. \tag{15}$$

The coefficients $Z_{k'k}$ are the elements of the supermatrix $\boldsymbol{Z}$. The eigenvalue problem in Equation (13) can be rewritten in a more general matrix form

$$\boldsymbol{ZA} = \boldsymbol{AD} \,, \tag{16}$$

where $\boldsymbol{D}$ is a diagonal matrix with entries $d_j = (\omega_1^2)_j$ for $j \in K$. Each column of $\boldsymbol{A}$ represents an eigenvector $\boldsymbol{c}_j$ composed of the expansion coefficients $\{c_{jk}\}_{k \in K}$.

2.3. General Matrix Element of a Meridional Flow

The general matrix element defined in Equation (15) depends on the perturbation operator $\mathcal{L}_1$. It describes the coupling strength between modes $k = (n, l, m)$ and $k' = (n', l', m')$ due to a velocity field $\boldsymbol{u}$. For a meridional flow, the operator is identified by the advection of a mode given by

$$\mathcal{L}_1 \boldsymbol{\xi}_k^0 = -2i\omega_k \rho_0 (\boldsymbol{u} \cdot \nabla)\boldsymbol{\xi}_k^0 . \quad (17)$$

In the following, we describe a meridional flow model and derive the general matrix element in terms of the flow model.

2.3.1. Description of a Meridional Flow

In spherical coordinates, the velocity field $\mathbf{u}$ of a meridional flow can be expanded in vector spherical harmonics of different degree s. As the meridional velocity field is described by a zonal poloidal flow which is symmetric in longitude ϕ, it has only contributions with azimuthal order 0, and one finds

$$\mathbf{u}(r,\theta,\phi) = \sum_s \left[u_s^0(r)\, Y_s^0(\theta,\phi)\, \boldsymbol{e_r} + v_s^0(r)\partial_\theta Y_s^0(\theta,\phi)\, \boldsymbol{e_\theta}\right] . \quad (18)$$

The expansion coefficients $u_s^0(r)$ and $v_s^0(r)$ of the corresponding radial and horizontal components are the flow strengths as a function of depth r. Assuming a *stationary* and *mass conserving* flow, described by the anelastic condition $[\nabla \cdot (\rho_0 \boldsymbol{u}) = 0]$, it is possible to express the horizontal flow v_s^0 in terms of u_s^0 via

$$\rho_0 r s(s+1) v_s^0 = \partial_r\left(r^2 \rho_0 u_s^0\right) . \quad (19)$$

2.3.2. The General Matrix Element

Inserting Equation (17) with Equation (18) into Equation (15), the general matrix element can be separated into an integration over the radial part dr and the angular part $d\Omega$ with solid angle Ω. The integration over $d\Omega$ determines the coupling of angular momenta of two modes via the flow. It can be expressed by the Wigner-3j symbol

$$\begin{pmatrix} l' & s & l \\ -m' & 0 & m \end{pmatrix} . \quad (20)$$

For a detailed description of its properties, see Edmonds (1974) and Lavely & Ritzwoller (1992). A non-vanishing contribution from the integral over longitude ϕ is obtained only for modes with equal azimuthal order $m = m'$. The coupling of angular momenta is further restricted by certain selection rules for l, l', and s as discussed in Roth & Stix (2008). Expressions for the general matrix elements generated by a meridional flow were derived by Lavely & Ritzwoller (1992) and Roth & Stix (2008) to

$$H_{k'k} = i\delta_{m\,m'}\omega_{\mathrm{ref}} \sum_{s=0}^{\infty} \left[(-1)^m \begin{pmatrix} l' & s & l \\ -m' & 0 & m \end{pmatrix} \times \int_0^R \rho_0(r) r^2 K_s^{k'k}(r) u_s^0(r) dr\right] , \quad (21)$$

with the poloidal flow kernel $K_s^{k'k}$ given in the Appendix. As argued in the Appendix, the flow kernels are purely real valued and from Equation (A1) one finds that the general matrix element is purely imaginary and $H_{k'k}^* = H_{kk'}$, i.e., that the general matrix $\boldsymbol{H}$ and therefore the supermatrix $\boldsymbol{Z}$ are both Hermitian.

2.4. Expansion of the General Matrix Elements and b-coefficients

The dependency of the general matrix element on azimuthal order m is restricted to the angular part condensed in the Wigner-3j symbol. The poloidal flow kernels actually do not depend on the azimuthal order, i.e., $K_s^{k'k} = K_s^{n'l',nl}$. Using the orthogonality properties of the Wigner-3j symbol (Edmonds 1974), it is possible to expand the general matrix element in Equation (21) in orthogonal polynomials in m:

$$H_{k'k} =: H_{n'l',nl}(m) = i\,\omega_{\mathrm{ref}} \sum_s b_{n'l',nl}^s \mathcal{P}_{l'l}^s(m) , \quad (22)$$

where we define the polynomials by

$$\mathcal{P}_{l'l}^s(m) = (-1)^{-m} \begin{pmatrix} l' & s & l \\ -m & 0 & m \end{pmatrix} \quad (23)$$

and introduce the b-coefficients

$$b_{n'l',nl}^s = \int_0^R \rho_0(r) r^2 K_s^{n'l',ln}(r) u_s^0(r) dr . \quad (24)$$

The polynomials $\mathcal{P}_{l'l}^s$ are analogously defined to the polynomials used to describe frequency splittings due to differential rotation (Ritzwoller & Lavely 1991). But as these frequency splittings are described by self coupling of modes with equal harmonic degree and azimuthal order, a more general definition for the polynomials must be used in the case of the meridional flow. However, this generalization of the polynomials has no influence on their orthogonality properties. The polynomials $\mathcal{P}_{l'l}^s$ are defined on a discrete interval $-\min(l,l') \leqslant m \leqslant \min(l,l')$ where they form an orthogonal and complete set of basis functions for fixed l, l' with respect to s. Due to the symmetry properties of the Wigner-3j symbol (Edmonds 1974), one can show that the polynomials also satisfy the symmetry relation:

$$\mathcal{P}_{l'l}^s(-m) = (-1)^{l'+s+l} \mathcal{P}_{l'l}^s(m) . \quad (25)$$

As the poloidal flow kernel in Equation (A1) is only non-vanishing if $l' + s + l$ is even it thus follows

$$H_{n'l',nl}(-m) = H_{n'l',nl}(m) . \quad (26)$$

Therefore, the general matrix elements for fixed (n, l) and (n', l') are symmetric with respect to azimuthal order m. The b-coefficients are independent of m and relate the general matrix elements linearly to the radial meridional flow strength u_s^0 with corresponding degree s.

3. APPROXIMATION OF THE EIGENVECTORS OF THE SUPERMATRIX

Lavely & Ritzwoller (1992) and Roth & Stix (2008) focused on the perturbation of mode frequencies or, respectively, the eigenvalues of the supermatrix in Equation (16) in the presence of a flow. Based on their work, we now regard the perturbation of the eigenfunctions. We derive approximate expressions for the eigenvectors $\boldsymbol{c}$ of the supermatrix when adapted to a meridional flow. The properties of the general matrix elements for a meridional flow allow us to make explicit use of the connection between quasi-degenerate and non-degenerate perturbation theory. We define coupling ratios which quantify the relative distortion of the eigenfunction of a mode with respect to the unperturbed eigenfunctions and compare the effects of a meridional flow on the eigenfunctions and on the eigenfrequencies.

According to Equation (11), the eigenvectors $\boldsymbol{c}$ are composed of the expansion coefficients of the perturbed eigenfunction in zeroth order. We regard a subset K of modes for which the quasi-degeneracy condition holds. As for a meridional flow only modes of equal azimuthal order m are able to couple, the supermatrix $\boldsymbol{Z}$ associated with K can be arranged into a block diagonal matrix where each block $\boldsymbol{Z}_m$ on the main diagonal belongs to a different azimuthal order m.

The set $K = \{j_1, \ldots, j_N\}$ shall be now restricted to modes of order m. We focus on a single mode $(n, l, m) \in K$ with frequency next or equal to ω_{ref} and denote it w.l.o.g. $j_1 = (n, l, m)$. We regard the perturbation of j_1 due to the presence of the other modes in K. For ease of notation we rename the indices in the following by setting $j_k = k$. Then the block-matrix $\boldsymbol{Z}_m$ for azimuthal order m is of the form

$$\boldsymbol{Z}_m = \begin{pmatrix} \lambda_1^0 & H_{12} & H_{13} & \cdots & H_{1N} \\ H_{21} & \lambda_2^0 & H_{23} & & H_{2N} \\ H_{31} & H_{32} & \lambda_3^0 & & H_{3N} \\ \vdots & & & \ddots & \vdots \\ H_{N1} & H_{N2} & H_{N3} & \cdots & \lambda_N^0 \end{pmatrix}. \tag{27}$$

The diagonal elements are determined by the squared frequency differences $\lambda_k^0 = \omega_k^2 - \omega_{\text{ref}}^2$. We use again an auxiliary parameter $0 \leqslant \epsilon \leqslant 1$ for the perturbation order and regard the eigenvalue problem of

$$\boldsymbol{D} + \epsilon \boldsymbol{H} = \begin{pmatrix} \lambda_1^0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^0 & 0 & & 0 \\ 0 & 0 & \lambda_3^0 & & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & & \lambda_N^0 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & H_{12} & H_{13} & \cdots & H_{1\,N} \\ H_{21} & 0 & H_{23} & & H_{2\,N} \\ H_{31} & H_{32} & 0 & & H_{3\,N} \\ \vdots & & & \ddots & \\ H_{N\,1} & H_{N\,2} & H_{N\,3} & & 0 \end{pmatrix}. \tag{28}$$

The matrix $\boldsymbol{H}$ is regarded as a perturbation of the diagonal matrix $\boldsymbol{D}$. The unperturbed matrix $\boldsymbol{D}$ has eigenvalues λ_k^0 and orthonormal eigenvectors which we identify by the standard basis vectors $\boldsymbol{v}_k^0 = (0, \ldots, 0, 1, 0, \ldots, 0)$ where the non-zero entry is at position k.

The unperturbed eigenvalues $\{\lambda_k^0\}$ are determined by the eigenfrequencies of different multiplets in K. We assume that they are non-degenerate and exploit the concepts of time-independent non-degenerate perturbation analysis (cf. Sakurai 1994) to derive approximations for the perturbed eigenvalues and eigenvectors of $\boldsymbol{Z}_m$. The perturbation of the eigenvalues up to second order $O(\epsilon^2)$ is

$$\lambda_k = \lambda_k^0 + \epsilon \left(\boldsymbol{v}_k^0\right)^t \boldsymbol{H}\, \boldsymbol{v}_k^0 + \epsilon^2 \sum_{i \in K, i \neq k} \frac{\left|\left(\boldsymbol{v}_i^0\right)^t \boldsymbol{H}\, \boldsymbol{v}_k^0\right|^2}{\lambda_k^0 - \lambda_i^0}. \tag{29}$$

For the non-degenerate perturbation of the eigenvectors up to second order, one finds

$$\begin{aligned} \boldsymbol{v}_k = \boldsymbol{v}_k^0 &+ \epsilon \sum_{i \in K, i \neq k} \left(\frac{\left(\boldsymbol{v}_i^0\right)^t \boldsymbol{H}\, \boldsymbol{v}_k^0}{\lambda_k^0 - \lambda_i^0} \right) \boldsymbol{v}_i^0 \\ &+ \epsilon^2 \left\{ \sum_{i \neq k} \sum_{j \neq k} \left(\frac{\left(\boldsymbol{v}_i^0\right)^t \boldsymbol{H}\, \boldsymbol{v}_j^0 \left(\boldsymbol{v}_j^0\right)^t \boldsymbol{H}\, \boldsymbol{v}_k^0}{\left(\lambda_k^0 - \lambda_i^0\right)\left(\lambda_k^0 - \lambda_j^0\right)} \right) \boldsymbol{v}_i^0 \right. \\ &\left. - \sum_{i \neq k} \left(\frac{\left(\boldsymbol{v}_i^0\right)^t \boldsymbol{H}\, \boldsymbol{v}_k^0 \left(\boldsymbol{v}_k^0\right)^t \boldsymbol{H}\, \boldsymbol{v}_k^0}{\left(\lambda_k^0 - \lambda_i^0\right)^2} \right) \boldsymbol{v}_i^0 \right\}, \end{aligned} \tag{30}$$

where we have used the conventional normalization $\boldsymbol{v}_k \boldsymbol{v}_k^0 = 1$ as the eigenvector is invariant up to multiplication with a factor. By setting $\epsilon = 1$, evaluating the expressions $(\boldsymbol{v}_i^0)^t \boldsymbol{H}\, \boldsymbol{v}_k^0$, and substituting λ_k^0 with the difference of the squared frequencies, Equation (29) evaluated for $k = 1$ simplifies to

$$\lambda_1 = \omega_1^2 - \omega_{\text{ref}}^2 + \sum_{i=2}^{N} \frac{|H_{i\,1}|^2}{\omega_1^2 - \omega_i^2}. \tag{31}$$

As the main diagonal elements of $\boldsymbol{H}$ are zero there are no contributions to the perturbed eigenvalues of first order in ϵ in Equation (31). The perturbation of the eigenvalues is of second order in ϵ.

For the perturbation of the first eigenvector $\boldsymbol{v}_1^0$ one finds from Equation (30)

$$\begin{aligned} \boldsymbol{v}_1 = \boldsymbol{v}_1^0 &+ \sum_{i=2}^{N} \left(\frac{H_{i\,1}}{\omega_1^2 - \omega_i^2} \right) \boldsymbol{v}_i^0 \\ &+ \sum_{i=2}^{N} \sum_{j=2}^{N} \left(\frac{H_{ij} H_{j1}}{\left(\omega_1^2 - \omega_i^2\right)\left(\omega_1^2 - \omega_j^2\right)} \right) \boldsymbol{v}_i^0, \end{aligned} \tag{32}$$

where the last expression for the contribution in second order of Equation (30) vanishes as $\boldsymbol{H}$ has zero main diagonal elements. The eigenvector $\boldsymbol{v}_1$ of $\boldsymbol{Z}_m$ depends up to first-order perturbation only on direct couplings between j_1 and j_i represented by general matrix elements H_{i1}. Couplings between modes $i, j \in K \backslash \{j_1\}$ assigned by matrix elements H_{ij} outside of the first row and column of $\boldsymbol{Z}_m$ can be neglected in first order. These matrix elements influence $\boldsymbol{v}_1$ in higher orders via indirect couplings. In second order an indirect coupling between two modes $i \rightarrow k$ is given by expressions $H_{ij} H_{jk}$ which represents the indirect coupling via the direct couplings $i \rightarrow j$ and $j \rightarrow k$. Therefore, up to first order, indirect couplings do not influence the perturbed eigenvectors.

3.1. Coupling Ratios and Frequency Shifts

We now regard the eigenvector perturbation in Equation (32) up to first order. The set of eigenvectors $\{\boldsymbol{v}_k^0\}_{k \in K}$ is related to the set of eigenfunctions $\{\boldsymbol{\xi}_k^0\}_{k \in K}$ by an isometric mapping. Therefore comparing the approximated coefficients of the eigenvector of $\boldsymbol{Z}_m$ up to first order with that of the corresponding perturbed eigenfunction of zeroth order in Equation (11), we find the identity

$$c_{jk} = \begin{cases} 1 & \text{for} \quad k = j \\ \frac{H_{kj}}{\omega_j^2 - \omega_k^2} & \text{for} \quad k \in K \backslash \{j\} \end{cases}. \tag{33}$$

The perturbed eigenfunction can be written as

$$\boldsymbol{\xi}_j \approx \boldsymbol{\xi}_j^0 + \sum_{k\in K\setminus\{j\}} \left(\frac{H_{kj}}{\omega_j^2 - \omega_k^2} \right) \boldsymbol{\xi}_k^0 . \quad (34)$$

The result reflects the compromise between quasi-degenerate and non-degenerate perturbation theory. The zeroth-order eigenfunction is formally equal to the perturbed eigenfunction up to the first order of non-degenerate perturbation theory except that the summation is restricted to contributions from the subset K.

The eigenvectors $\boldsymbol{c}$ of the supermatrix derived above are normalized such that $c_{jj} = 1$. Other normalizations of $\boldsymbol{c}$ might be used, e.g., $\sum_{k\in K} |c_{jk}|^2 = 1$. However, in the next section we will see that the normalization is not relevant and we introduce the *coupling ratio* defined as

$$C_{jk} := c_{jk}/c_{jj} . \quad (35)$$

This ratio is defined so that it is independent of the chosen normalization of $\boldsymbol{c}$. In particular, for our previous normalization of $c_{jj} = 1$ the coupling ratio is equal to the expansion coefficients c_{jk}.

We compare the perturbation of the eigenfunctions with the perturbation of the eigenfrequencies. According to Lavely & Ritzwoller (1992), the frequency shift $\delta\omega_j$ of a mode j is defined such that $\omega_j \approx \omega_{\mathrm{ref}} + \delta\omega_j$ and can be approximated as

$$\delta\omega_j \approx \frac{1}{2\,\omega_{\mathrm{ref}}} \left(\omega_1^2\right)_j . \quad (36)$$

From Equation (31), we obtain for the frequency shift

$$\delta\omega_j \approx \frac{1}{2\,\omega_{\mathrm{ref}}} \sum_{k\in K\setminus\{j\}} \frac{|H_{kj}|^2}{\omega_j^2 - \omega_k^2} , \quad (37)$$

which is in accordance with the results of Roth & Stix (2008) and Chatterjee & Antia (2009). Equation (37) shows that the frequency shift is in second order of the coupling strength given by H_{kj}. Therefore, the frequency shift does not provide information about the sign of the flow as it is not preserved in this quantity. Furthermore, the frequency shift is the sum of contributions from different coupling partners that cannot be resolved. This complicates the inference of the flow on this quantity. However, the perturbation of the eigenfunctions given in Equation (34) is linear in the coupling strengths which preserves the sign of the flow. Moreover, the single contributions to the perturbed eigenfunction are orthogonal to each other. This orthogonality can be exploited to resolve for the single contributions to the perturbation as shown in the following section.

4. GLOBAL SOLAR OSCILLATIONS AND AMPLITUDE RATIOS

The coupling ratio defined in Section 3 is a measure for the perturbation of an eigenfunction due to a meridional flow. We now relate this quantity to the observable properties of solar p-modes as obtained from Doppler velocity field measurements. For this purpose, we define a new observable, the *amplitude ratio*, and derive a set of linear equations which relate the coupling ratios with the observable amplitude ratios. Further, a rough estimate for the expected magnitude of the observables due to an exemplary meridional flow on the observables is presented.

The properties of solar p-modes can be investigated from the global oscillation time series $o_{lm}(t)$ for distinct degrees l and orders m. They are derived from spatiotemporal measurements of the solar Doppler velocity field $\boldsymbol{v}(R, \theta, \phi, t)$ observed in the photosphere at depth R. The velocity field is decomposed into spherical harmonics (Schou & Brown 1994)

$$o_{l'm'}(t) = \int W(\theta,\phi)\overline{Y_{l'}^{m'}}(\theta,\phi)\boldsymbol{v}(R,\theta,\phi,t) \cdot \boldsymbol{e}(\theta,\phi)\, r^2 \sin\theta\, d\theta\, d\phi . \quad (38)$$

The weighting function $W(\theta, \phi)$ is a spatial apodization function counting for the restricted visibility of the Doppler field at the solar surface by the instrument. The vector $\boldsymbol{e}$ is the line-of-sight unit vector between observer and the Sun.

Following Equation (34), the contribution of a perturbed eigenfunction $\boldsymbol{\xi}_k$ with corresponding mode coupling set K_k to the Doppler velocity field $\boldsymbol{v}$ is

$$\boldsymbol{v}_k = \alpha_k(t) \sum_{k''\in K_k} c_{kk''} \boldsymbol{\xi}_{k''}^0 , \quad (39)$$

where $\alpha_k(t)$ is the oscillation amplitude with perturbed frequency $\tilde{\omega}_k$. Inserting $\boldsymbol{v} = \sum_k \boldsymbol{v}_k$ with Equation (39) into Equation (38), one finds for an observable global oscillation l', m' the expression

$$o_{l'm'}(t) = \sum_k \alpha_k(t) \sum_{k''\in K_k} c_{kk''} \xi_{k''}^r(R) L_{k'k''} . \quad (40)$$

The factors

$$\xi_{k''}^r(R) L_{k'k''} = \int W\, \overline{Y_{l'}^{m'}} \left[\xi_{n''l''}^r(R) Y_{l''}^{m''} \boldsymbol{e_r} + \xi_{n''l''}^h(R) \nabla_h Y_{l''}^{m''} \right] \cdot \boldsymbol{e}\, r^2\, d\Omega \quad (41)$$

refer to the leakage due to imperfect observation of the global oscillation velocity field (Schou & Brown 1994; Korzennik et al. 2004). Applying the Fourier transform on both sides of Equation (40), a corresponding expression for the global oscillation is obtained in the frequency domain

$$\tilde{o}_{l'm'}(\omega) = \sum_k \tilde{\alpha}_k(\omega) \sum_{k''\in K_k} c_{kk''} \xi_{k''}^r(R) L_{k'k''} \in \mathbb{C} , \quad (42)$$

where $\tilde{\alpha}_k(\omega)$ is the Fourier transform of $\alpha_k(t)$. The coefficients $c_{kk''}$ are non-zero only if $k, k'' \in K_k$. Thus, there is a contribution of a mode k to a global oscillation $o_{l'm'}$ only if $L_{k'k''} c_{kk''} \neq 0$ for all $k'' \in K_k$. If there was no meridional flow there is also no coupling, i.e., $c_{kk''} = 0$ for all k and $k'' \neq k$, and one obtains

$$\tilde{o}_{l'm'}(\omega) = \sum_k \tilde{\alpha}_k(\omega) \xi_k^r(R) L_{k'k} \in \mathbb{C} . \quad (43)$$

4.1. Amplitude Ratios

We regard a mode $k = (n, l, m)$ with frequency $\tilde{\omega}_k$ and its coupling partners in K_k. We build the fraction between the Fourier amplitudes of global oscillations $o_{l,m}$ and $o_{l',m}$ corresponding to the modes in K_k. We evaluate this fraction at the frequency $\tilde{\omega}_k$ and define the *amplitude ratio*

$$y_{lm\,l'm}(\tilde{\omega}_k) := \frac{\tilde{o}_{l'm}(\tilde{\omega}_k)}{\tilde{o}_{lm}(\tilde{\omega}_k)} \in \mathbb{C} \quad (44)$$
$$= \frac{\tilde{\alpha}_k(\tilde{\omega}_k)\sum_{k''\in K_k} c_{kk''}\xi_{k''}^r(R)L_{k'k''} + \sum_{k'''\neq k} \tilde{\alpha}_{k'''}(\tilde{\omega}_k)\sum_{k''\in K_{k'''}} c_{k'''k''}\xi_{k''}^r(R)L_{k'k''}}{\tilde{\alpha}_k(\tilde{\omega}_k)\sum_{k''\in K_k} c_{kk''}\xi_{k''}^r(R)L_{kk''} + \sum_{k'''\neq k} \tilde{\alpha}_{k'''}(\tilde{\omega}_k)\sum_{k''\in K_{k'''}} c_{k'''k''}\xi_{k''}^r(R)L_{kk''}} .$$

The amplitude ratios $y_{lm\,l'm}(\tilde{\omega}_k)$ relate the observable global oscillations with the coefficients $c_{kk'}$. For each mode we may assume that the spectral bandwidth of its oscillation amplitude due to the damping of the mode is small and set $\tilde{\alpha}_{k'''}(\tilde{\omega}_k) \approx \tilde{\alpha}_k(\tilde{\omega}_k)\delta_{kk'''}$. This assumption is a good approximation at least for modes with small eigenfrequencies as they typically have small damping rates. Under this assumption, the oscillation frequencies of different modes separate in the frequency domain and we may discard in Equation (44) the sums along k'''.

We regard the relation between the amplitude ratio and the coupling ratios in Equation (44) for the case of a perfect observation of the radial component of the velocity field over the total solar surface. Therefore, we set $W = 1$ and $\boldsymbol{e} = \boldsymbol{e}_r$. Then, the leakage effect vanishes as from the orthogonality property of the spherical harmonics it follows $\xi^r_{k''}(R)L_{k'k''} = \delta_{l'l''}\delta_{m'm''}\xi^r_{n''l'}(R)$. We note that the set of coupling modes K_k in Equation (44) is determined by the quasi-degeneracy condition applied to the unperturbed eigenfrequencies and thus depends actually only on radial order and harmonic degree. For the case of a perfect observation, the relation between the coupling ratio and the amplitude ratio simplifies notably if we make the following assumption: each mode k'' in the coupling set K_k is uniquely identified by its harmonic degree l'' alone, i.e., two modes of equal harmonic degree but different radial order are not in K_k. We refer to this as *assumption A* in the text and comment on it in the discussion. Given *assumption A*, Equation (44) simplifies for all $k' \in K_k$ to

$$y_{lm\,l'm}(\tilde{\omega}_k) = \frac{\xi^r_{k'}(R)}{\xi^r_k(R)}\frac{c_{kk'}}{c_{kk}} = \frac{\xi^r_{k'}(R)}{\xi^r_k(R)}C_{kk'}\,, \tag{45}$$

i.e., the amplitude ratio is then proportional to the coupling ratio. We note that if *assumption A* is not fulfilled then the amplitude ratio is proportional to a sum of weighted coupling ratios as the summation index k'' in Equation (44) is running over different radial orders.

4.2. Linear Equations for the Coupling Ratios

Given that the leakage coefficients $L_{k'k''}$ are known, it is possible to derive a system of linear equations for the coupling ratios. Therefore, we divide the numerator and denominator in Equation (44) by $\xi^r_k(R)\,c_{kk} \neq 0$ and introduce the *weighted coupling ratio*

$$x_{kk''} := \frac{\xi^r_{k''}(R)}{\xi^r_k(R)}\frac{c_{kk''}}{c_{kk}} = \frac{\xi^r_{k''}(R)}{\xi^r_k(R)}C_{kk''} \in \mathbb{C}\,,\ k'' \in K_k\,. \tag{46}$$

We substitute the ratios in Equation (44) for $x_{kk''}$ and obtain the linear equation

$$y_{kk'}(\tilde{\omega}_k) = \frac{\sum_{k''\in K_k} x_{kk''}L_{k'k''}}{\sum_{k''\in K_k} x_{kk''}L_{kk''}}\,,\quad \forall\, k' \in K_k\,. \tag{47}$$

Given that the previously made *assumption A* holds we find $|K_k|$ equations with $|K_k|$ unknown variables $x_{kk'}$. Using $x_{kk} = 1$ (see Equation (46)) the linear equations can be rewritten to a set of $|K_k| - 1$ linear equations:

$$L_{k'k} - y_{kk'}(\tilde{\omega}_k)L_{kk} = -\sum_{k''\in K_k\backslash k} x_{kk''}(L_{k'k''} - y_{kk'}(\tilde{\omega}_k)L_{kk''}), \tag{48}$$

with $k' \in K_k\backslash k$ and $|K_k| - 1$ unknowns. It can be written in matrix form

$$z_k = -\boldsymbol{B}\boldsymbol{x}_k\,, \tag{49}$$

with

$$z_{kk'} = L_{k'k} - y_{kk'}(\tilde{\omega}_k)L_{kk} \tag{50}$$

$$B_{k'k''} = L_{k'k''} - y_{kk'}(\tilde{\omega}_k)L_{kk''}\,. \tag{51}$$

Solving this linear equation for $\boldsymbol{x}_k$, one obtains the coupling ratios $C_{kk''}$ belonging to a perturbed mode k. These equations relate the observable amplitude ratios to the perturbation of the eigenfunctions due to a meridional flow taking into account the effect of leakage.

4.3. Magnitude of the Amplitude Ratios

We investigate the order of magnitude of amplitude ratios due to a meridional flow. For this aim a forward modeling is necessary as shown in Schad et al. (2011). For simplicity, we assume that the meridional flow consists of a single $s = 2$ component which represents one flow cell on each hemisphere. We further assume that the flow cells are confined approximately between the surface R and the bottom of the convection zone $r_b \approx 0.7R$. We assume a maximum poleward horizontal flow strength at the surface of about 10 m s^{-1} by setting $v_2(R) = 10$ m s^{-1}. For the radial flow profile in the region $r_b \leqslant r \leqslant R$, we use the simple model given in Roth & Stix (2008):

$$u_2(r) = u_0 \sin\left(\frac{r - r_b}{R - r_b}\pi\right)\,, \tag{52}$$

with vanishing flow strength elsewhere. From Equation (19) we deduce $u_0 \approx -0.6\,v_2(R)$.

The magnitude of the coupling ratios depends on the degree of the modes. Exemplarily, we choose the mode $k = (l = 100,\ n = 6,\ m = 0)$. According to the selection rules given in Roth & Stix (2008), its neighboring coupling modes forming the set $K_{(l=100,\,n=6,\,m=0)}$ are given by the modes $l' = 98$ and $l' = 102$ with $n' = 6$ and $m' = 0$.

Using eigenfunctions and eigenfrequencies obtained by Model S of Christensen-Dalsgaard et al. (1996), we obtain for the coupling ratio values of $C_{kk''} \approx 0.03\,i$ for both k'' with $l'' = 98$ and $l'' = 102$. The magnitude for the coupling ratio is thus of the order of 1% which is also found for more complex flow models as shown in Schad et al. (2011).

The observable amplitude ratio is essentially influenced by the leakage matrix. Major contributions to the leakage matrix elements are given by the incomplete integration over the visible solar hemisphere, the projection onto the line of sight, and the spatial apodization function W of the observing instrument. As a rough approximation for the leakage matrix, we restrict the calculation to the radial component of the Doppler velocity field. For the apodization function W we use a simple cosine bell and incorporate the projection on the line of sight and the incomplete integration limits.

We regard the amplitude ratio between k and $k' = (n' = 6,\ l' = 102,\ m' = 0)$. The coupling ratios are nearly constant, i.e., we set $C_{kk''} \approx C = 0.03\,i$ as given above for all $k'' \in K_k$ and use $\xi^r_{k''}(R)/\xi^r_k(R) \approx 1$. We then obtain from Equation (47) for the amplitude ratio

$$y_{lm\,l'm}(\tilde{\omega}_k) \approx \frac{L_{k'k} + C\sum_{k''\in K_k\backslash\{k\}} L_{k'k''}}{L_{kk} + C\sum_{k''\in K_k\backslash\{k\}} L_{kk''}} \approx -0.5 + 0.015i\,. \tag{53}$$

The real part of the amplitude ratio is due to the leakage while the imaginary part is due to the presence of a flow. According to Equation (45) for the case of no leakage, one would find $y_{lm\,l'm}(\tilde{\omega}_k) \approx C$. Hence the leakage decreases the observability of the coupling ratios. For the leakage matrix and flow model used here, the decrease is given by a factor of two. The coupling ratio scales linearly with the maximum flow strength at the surface. Furthermore, if $C \ll L_{kk}/\sum_{k''\in K_k\setminus\{k\}} L_{kk''}$ one finds that the imaginary part of the amplitude ratio also scales approximately linear with the maximum flow strength $v_2(R)$, i.e., $\Im(y_{lm}^{l'm}) \propto v_2(R)$. Hence doubling the flow strength results in an amplitude ratio twice as large as that given here.

The rough estimations made here show that the precision for the estimation of the amplitude ratios from data must be better than 1% when inferring signatures of a meridional flow with strength of about 10 m s^{-1} at the surface.

5. SUMMARY

The amplitude ratios present a new helioseismic quantity. With the here used approximations they are related to the radial components u_s^0 of a meridional flow via the coupling ratios:

$$C_{kk'} = \frac{i\,\omega_{\text{ref}}}{\omega_k^2 - \omega_{k'}^2} \sum_{s=0}^{\infty} \left[(-1)^m \begin{pmatrix} l' & s & l \\ -m & 0 & m \end{pmatrix} \times \int_0^R \rho_0(r) r^2 K_s^{k'k}(r) u_s^0(r) dr \right]. \quad (54)$$

As shown in Equation (22), the coupling ratios can be expanded in orthogonal polynomials with respect to the azimuthal order m. From the respective expansion coefficients (b-coefficients), the inversion for the radial flow strengths $u_s^0(r)$ from global solar oscillations is now reduced to an integral equation. It may be solved by application of inversion methods like the Subtractive Optimally Localized Averages or a regularized least squares technique, e.g., used to infer the differential rotation (Pijpers & Thompson 1992, 1994; Schou et al. 1994). By means of the mass conservation law and the assumption of a divergence-free flow, the horizontal flow strengths $v_s(r)$ can be derived from the estimated radial flow strengths via Equation (19).

6. DISCUSSION AND CONCLUSION

In this paper, we present a coherent treatment of the forward and inverse problem for the effect of a stationary, divergence-free meridional flow on global solar oscillations. We focus on the perturbation of the p-mode eigenfunctions and derive a formalism useful for measuring the meridional flow as a function of latitude and radius inside the Sun.

The perturbation analysis shows that the perturbation of the eigenfrequencies due to a meridional flow is of second order in the flow strength. The quasi-degenerate perturbation of the eigenvectors in zeroth order, however, corresponds to a non-degenerate perturbation of the eigenvector in first order restricted to a subset of modes. As the eigenfunction perturbation depends linearly on the flow strength, the sign of the flow is preserved in the observable quantities, i.e., the amplitude ratios. For the frequency shifts the sign of the flow is not preserved. The amplitude ratios are expected in the order of 1% for a meridional circulation with a surface amplitude of about 10 m s^{-1}; they scale approximately linearly with the flow strength. Consequently, studying the eigenfunction perturbations offers a new possibility to assess the meridional flow.

In our derivations in Section 4, we assumed that each mode in a set of coupling modes K can be uniquely identified by its harmonic degree; see *assumption A*. This assumption is necessary for the solvability of the set of linear equations for the coupling ratios (see Equation (49)). If the assumption is not fulfilled the set of linear equations is underdetermined and cannot be solved. The validity of this assumption depends on the quasi-degeneracy condition $|\omega_{\text{ref}}^2 - \omega_k^2| \leqslant \delta\omega^2$. The choice of $\delta\omega$ might be motivated by the rapid decrease of the coupling strength with an increase of the difference of the squared eigenfrequencies (Lavely & Ritzwoller 1992) as mirrored in Equation (54). Strong mode couplings are expected only for small values of $\delta\omega$. For the Sun, the large frequency separation $\Delta\nu = 136\,\mu$Hz between modes of equal harmonic degree and adjacent radial order might be regarded as an upper limit for $\delta\omega$ if adjacent modes do not contribute to the set of coupling modes. We tested the assumption using Model S of Christensen-Dalsgaard et al. (1996) for modes with degrees $0 \leqslant l \leqslant 200$ and for a meridional flow with degrees $s = 1, \ldots, 10$. Applying the selection rules for mode coupling, one finds that *assumption A* is fulfilled for all possible mode couplings if $\delta\omega \leqslant 2\pi\, 66\,\mu$Hz, which is about half of the large frequency separation. Relaxing the quasi-degeneracy condition by using, e.g., $\delta\omega = 2\pi\, 70\,\mu$Hz, one finds that the assumption is fulfilled nearly for all sets of allowed mode couplings except for a few mode coupling sets covering modes of small degrees $l \leqslant 17$. As a strategy we propose to omit such mode coupling sets from an inversion analysis for a meridional flow. This may reduce the resolvability of the flow especially at large depths. However, in the tested frequency range for mode couplings only a few coupling sets would have to be omitted and we thus would expect only a small decrease of resolution in depth.

A crucial point in our presented investigations is given by the leakage matrix. The leakage matrix may have similar characteristics as the meridional flow and therefore it is very important to have accurate knowledge about the leakage matrix elements. Formally, the leakage matrix is multiplied with the eigenvectors $\boldsymbol{c}$ of the coupling coefficients. Misspecification of the leakage matrix may result in biased estimates of the meridional flow. This problem needs to be addressed separately.

The amplitude ratio introduced here provides a new helioseismic measurement quantity. It relates the observable global oscillations with the meridional flow. Thus, our new concept presented here allows to probe for the meridional flow for the first time from global helioseismic measurements. As global p-modes are penetrating the Sun down to its core we conclude that this concept has the potential to infer the meridional flow throughout the solar interior even at large depths.

This work was supported by the Deutsche Forschungsgemeinschaft DFG, grant Ti315/4-2. A.S. and M.R. acknowledge support by the European Helio- and Asteroseismology Network (HELAS) which was funded as Coordination Action under the European Commission's Sixth Framework Programme (FP6).

APPENDIX

POLOIDAL FLOW KERNEL

The poloidal flow kernel $K_s^{k'k}$ appearing in Equation (21) is given by

$$K_s^{k'k}(r) = 8\pi \gamma_{l'}\gamma_l\gamma_s \left\{ \tilde{R}_s^{k'k}(r) - \frac{1}{s(s+1)}\frac{d}{dr}\left(r\tilde{H}_s^{k'k}(r)\right)\right\}. \quad (A1)$$

The kernels $\tilde{R}_s^{k'k}$ and $\tilde{H}_s^{k'k}$ are derived in Lavely & Ritzwoller (1992):

$$\tilde{R}_s^{k'k} = \frac{1}{2}\left\{\left(\xi_{k'}^r \frac{d}{dr}\xi_k^r - \left(\frac{d}{dr}\xi_{k'}^r\right)\xi_k^r\right) B_{l'sl}^0 + \left(\xi_{k'}^h \frac{d}{dr}\xi_k^h - \left(\frac{d}{dr}\xi_{k'}^h\right)\xi_k^h\right) B_{l'sl}^1\right\} \quad \text{(A2)}$$

and

$$\tilde{H}_s^{k'k} = \frac{1}{2r}[l(l+1) - l'(l'+1)]\left\{\xi_{k'}^r \xi_k^r B_{l'sl}^0 + \xi_{k'}^h \xi_k^h B_{l'sl}^1\right\} + \frac{1}{r}\left\{\xi_{k'}^h \xi_k^r B_{l'ls}^1 - \xi_{k'}^r \xi_k^h B_{ll's}^1\right\}, \quad \text{(A3)}$$

with

$$\gamma_s = \sqrt{\frac{2s+1}{4\pi}}, \quad \text{(A4)}$$

$$B_{l'sl}^0 = \frac{1}{2}(1 + (-1)^{l'+s+l})\begin{pmatrix} l' & s & l \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{(A5)}$$

$$B_{l'sl}^1 = \frac{1}{2}[l'(l'+1) + l(l+1) - s(s+1)]\, B_{l'sl}^0 . \quad \text{(A6)}$$

Due to the factor $(1 + (-1)^{l'+s+l})$ appearing in the expression for the poloidal flow kernels, the general matrix elements are vanishing for uneven parity of $l' + s + l$. The radial and horizontal eigenfunctions of the solar model are real valued as well as the Wigner-3j symbol. Consequently, the poloidal flow kernels are real valued and from Equations (A1)–(A6) it follows $K_s^{k'k} = -K_s^{kk'}$.

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