



DETECTING COUPLING DIRECTIONS IN MULTIVARIATE OSCILLATORY SYSTEMS

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Determination of synchronization phenomena between pairs of coupled multivariate processes is of particular interest in Nonlinear Dynamics. Besides synchronization phenomena, coupling directions between the processes are investigated. We present an approach to analyze coupling directions in multivariate oscillatory stochastic systems. We propose usage of partial directed coherence developed in the framework of linear stochastic processes. We show that partial directed coherence is also applicable to detect coupling directions in nonlinear systems such as coupled stochastic van der Pol and stochastic Rössler systems. Furthermore, a differentiation between direct and indirect couplings in multivariate systems is possible when applying partial directed coherence.

Keywords: Nonlinear dynamics; coupling directions; partial directed coherence.

1. Introduction

Analysis of multivariate systems has gained particular interest during the past years. To this aim, methods have been developed for linear stochastic systems as well as for nonlinear deterministic systems. These methods allow, for instance, the detection of interactions between the processes [Brockwell & Davis, 1998; Dahlhaus, 2000; Pikovsky *et al.*, 2001]. Extensions to the nonlinear stochastic case have also been addressed [Tass *et al.*, 1998].

Partial directed coherence has been developed in the framework of linear stochastic systems to detect the directions of interactions in multivariate systems [Baccala & Sameshima, 2001]. Moreover, partial directed coherence enables to distinguish

between direct and indirect interactions. The interaction network can thus be analyzed and determined completely.

Partial directed coherence is based on modeling the multivariate system by linear vector autoregressive processes. Therefore, it is not clear whether or not partial directed coherence can be readily applied to stochastic oscillatory systems. In this paper, we present an application of partial directed coherence to simulated data originating from stochastic oscillatory systems.

2. Partial Directed Coherence

We briefly introduce the concept of partial directed coherence in the following. Consider a

n -dimensional vector autoregressive process VAR[p] of model order p

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}(t) = \sum_{r=1}^p \mathbf{a}_r \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}(t-r) + \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}(t) \quad (1)$$

where η_i are the components of multivariate standard Gaussian distributed random variables and \mathbf{a}_r matrices with entries $a_{kl,r}$ quantifying the time delayed influences of the history of the process on the current value. Fourier transformation of the coefficient matrices

$$A_{kl}(\omega) = 1 - \sum_{r=1}^p a_{kl,r} e^{i\omega r} \quad (2)$$

leads to the definition of partial directed coherence

$$\pi_{k \leftarrow l}(\omega) = \frac{|A_{kl}(\omega)|}{\sqrt{\sum_{j=1}^n |A_{jl}(\omega)|^2}}. \quad (3)$$

The denominator guarantees that partial directed coherence is normalized in $[0, 1]$. A partial directed coherence value of zero indicates an absent directed interaction from process l to process k at frequency ω .

To estimate partial directed coherence, multivariate Yule–Walker equations can be utilized in order to estimate the VAR coefficient matrices. Direct usage of Eq. (3) leads to an estimate of partial directed coherence. Recently, an analytic significance level has been introduced to decide whether or not partial directed coherence is significantly different from zero when estimated for finite time series [Schelter et al., 2005]. This significance level has been shown to be frequency dependent. We suggest to chose the model order p by comparing the parametric auto-spectra with auto-spectra estimated by means of a nonparametric approach.

3. Model Systems Under Investigation

In the following, the two oscillatory systems are introduced and the application of partial directed coherence to these systems is reported.

3.1. Network of four coupled van der Pol oscillators

The first model system is a network of four coupled stochastic van der Pol oscillators [van der Pol, 1922]

$$\ddot{x}_i = \mu(1 - x_i^2)\dot{x}_i - \omega_i^2 x_i + \sigma \eta_i + \sum_{j \neq i}^4 \varepsilon_{ij}(x_j - x_i), \quad i = 1, \dots, 4 \quad (4)$$

with standard normally distributed noise η_i . The parameters have been chosen to be $\mu = 5$, $\omega_1 = 1.02$, $\omega_2 = 0.97$, $\omega_3 = 1.01$, $\omega_4 = 0.5$, and $\sigma = 1.5$. The couplings between the oscillators have been modeled by $\varepsilon_{12} = 0.3$, $\varepsilon_{14} = 0.25$, $\varepsilon_{24} = 0.25$, $\varepsilon_{41} = 0.3$, and $\varepsilon_{42} = 0.3$. The remaining coupling strengths have been chosen to be equal to zero. Note, that the stochasticity entering the differential equations can in general modify the characteristics of the model systems considerably.

A four-dimensional VAR[20] was fitted to the simulated van der Pol system with $N = 50.000$ data points for each component. The results of partial directed coherence analysis are shown in Fig. 1. The significance level is indicated by the red line. At the oscillation frequencies of the oscillators, the significance level obtains higher values preventing erroneous conclusions. The following coupling directions, summarized in the graph in Fig. 1, are observed by significant partial directed coherence values at the oscillation frequency: $X_2 \rightarrow X_1$, $X_4 \rightarrow X_1$, $X_4 \rightarrow X_2$, $X_1 \rightarrow X_4$, and $X_2 \rightarrow X_4$. The simulated coupling structure is reproduced correctly by partial coherence analysis.

3.2. Network of four-coupled Rössler oscillators

The second model system is a network of four-coupled Rössler oscillators [Rössler, 1976]

$$\begin{pmatrix} \dot{X}_j \\ \dot{Y}_j \\ \dot{Z}_j \end{pmatrix} = \begin{pmatrix} -\omega_j Y_j - Z_j + \left[\sum_i \varepsilon_{ji} (X_i - X_j) \right] + \sigma_j \eta_j \\ \omega_j X_j + a Y_j \\ b + (X_j - c) Z_j \end{pmatrix}, \quad i, j = 1, 2, 3, 4. \quad (5)$$

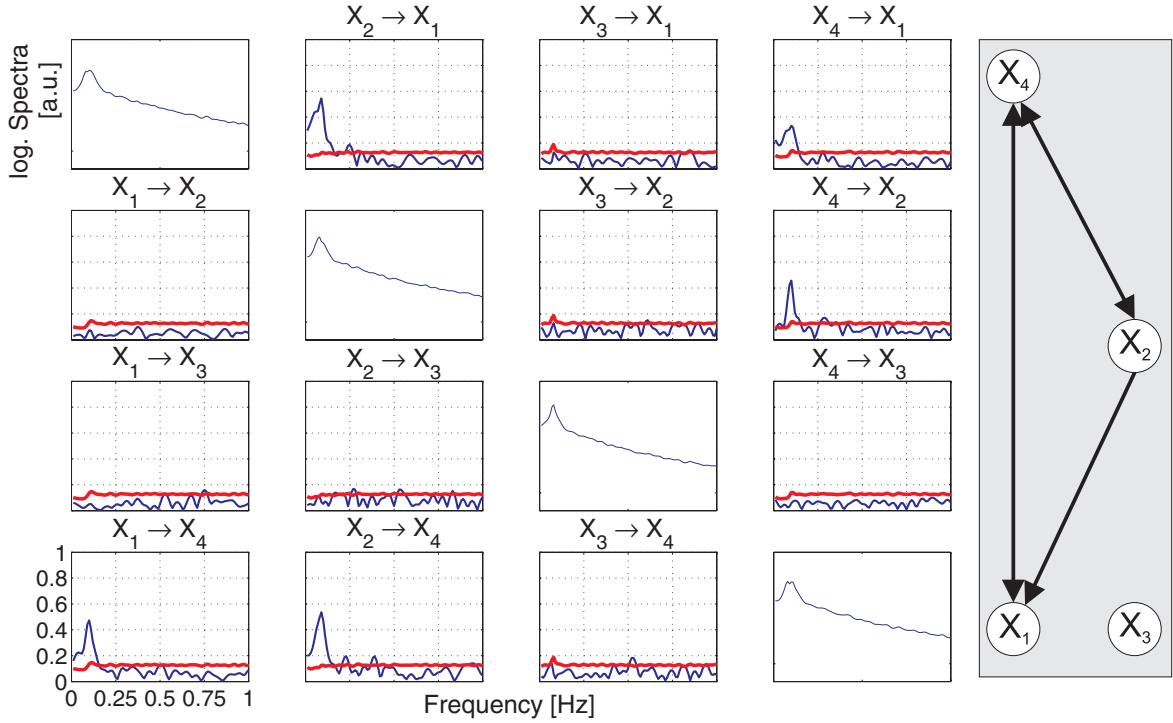


Fig. 1. Results of partial directed coherence analysis for a four-dimensional van der Pol system. On the diagonal the spectra of the four oscillators are shown and on the off-diagonal the partial directed coherences. The red curves indicate the frequency dependent 5%-significance levels. The graph summarizes the coupling scheme detected by partial directed coherence, which is in agreement with the coupling scheme simulated.

The parameters have been chosen to be $\sigma_j = 1.5$, $\omega_1 = 0.99$, $\omega_2 = 1.05$, $\omega_3 = 0.97$, $\omega_4 = 1.02$, $a = 0.15$, $b = 0.2$, $c = 10$; η_j denotes Gaussian distributed white noise. The couplings between the oscillators have been modeled by $\varepsilon_{12} = 0.05$, $\varepsilon_{13} = 0.04$, $\varepsilon_{23} = 0.05$, $\varepsilon_{24} = 0.05$, $\varepsilon_{31} = 0.06$, and $\varepsilon_{42} = 0.05$. The remaining coupling strengths have been chosen to be equal to zero. For each oscillator, $N = 50.000$ have been simulated. Partial directed coherence was applied to the four X -components of the Rössler oscillators utilizing a four-dimensional VAR[200]. The results are shown in Fig. 2. The following coupling directions, summarized in the graph in Fig. 2, are observed by significant partial directed coherence values at the oscillation frequency: $X_2 \rightarrow X_1$, $X_3 \rightarrow X_1$, $X_3 \rightarrow X_2$, $X_4 \rightarrow X_2$, $X_1 \rightarrow X_3$ and $X_2 \rightarrow X_4$. The coupling structure simulated is again reproduced correctly by partial coherence analysis.

3.3. Detection of coupling directions — A test for the power

To test the power of partial directed coherence for detection of coupling directions in nonlinear

stochastic systems, we have investigated a two-oscillator unidirectionally coupled Rössler system. The coupling strength ε_{12} was varied from 0.001 to 0.1, while the coupling in the opposite direction was absent ($\varepsilon_{21} = 0$). The parameters were chosen to be $\omega_1 = 1.015$, $\omega_2 = 0.985$, $a = 0.15$, $b = 0.2$, $c = 10$, and $\sigma_{1,2} = 1.5$. A two-dimensional VAR[200] was fitted to the simulated $N = 50.000$ data points for each X -component and each coupling strength.

In Fig. 3 the partial directed coherence values (blue curves) at the oscillation frequencies and the corresponding critical values for a 5%-significance level (red curves) are given in dependence on the coupling strength ε_{12} . The functional relationship of partial directed coherence and the critical value for the coupling direction from X_2 to X_1 at the frequency of oscillator X_1 are presented in (a), for the coupling direction from X_2 to X_1 at the frequency of oscillator X_2 in (b), for the coupling direction from X_1 to X_2 at the frequency of oscillator X_1 in (c), and for the coupling direction from X_1 to X_2 at the frequency of oscillator X_2 in (d). Since there is only a minor difference between the results for the different frequencies, we restrict the following interpretations to the frequency of oscillator X_1 .

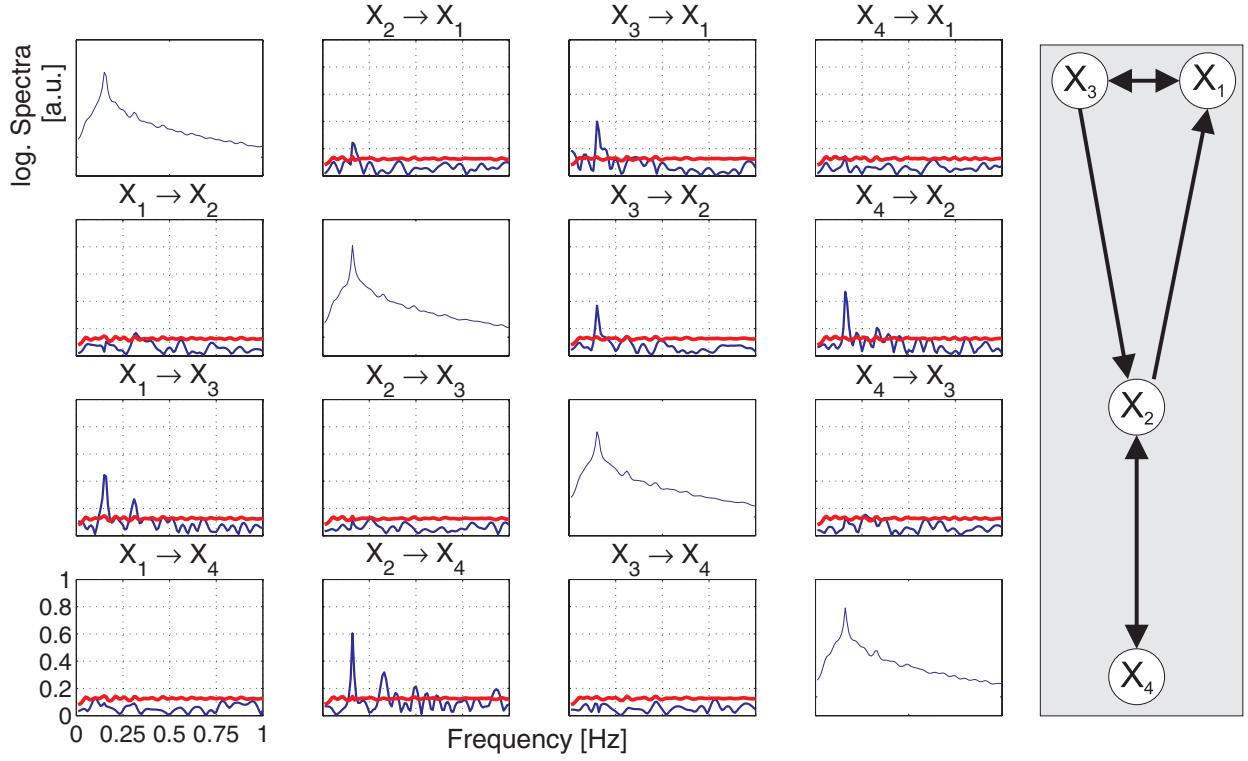


Fig. 2. Results of partial directed coherence analysis for a four-dimensional Rössler system. The analysis is performed on the X -components. On the diagonal the spectra of the four oscillators are given, while on the off-diagonal the partial directed coherences are shown. The red curves indicate the frequency dependent 5%-significance levels. The graph summarizes the detected coupling scheme by partial directed coherence. It is in agreement with the coupling scheme simulated.

The functional relation of partial directed coherence crosses the functional relationship of the critical value at $\varepsilon_{12} = 0.025$ for the direction $X_2 \rightarrow X_1$. For the opposite coupling direction,

partial directed coherence values are not significant. A detection of the correct coupling directions between stochastic Rössler oscillators is possible for coupling strengths higher than 0.025, which

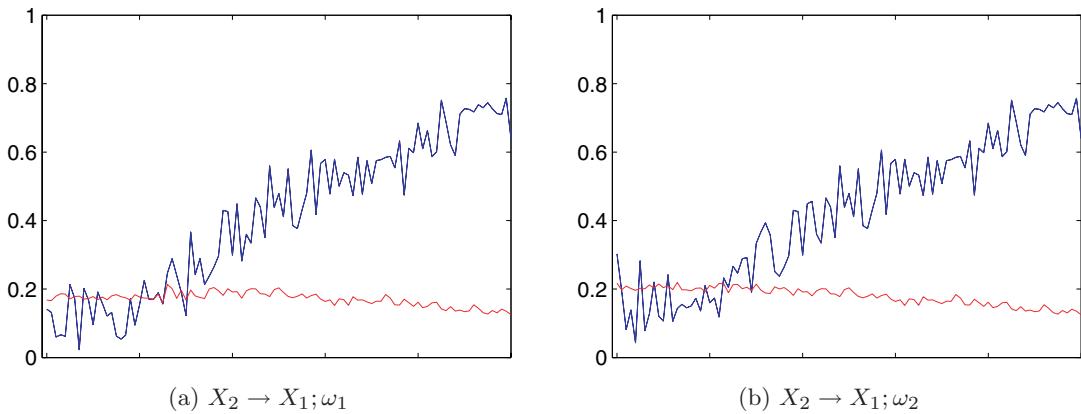


Fig. 3. To test the power of partial directed coherence in detecting coupling directions, the empirical functional relationship of partial directed coherences and the corresponding critical values in dependence of a unidirectional coupling ε_{12} have been investigated. Partial directed coherence as well as the corresponding critical value is evaluated at both oscillation frequencies of the oscillators X_1 and X_2 . Results for both frequencies are quite similar. The correct coupling direction is detected for coupling strengths $\varepsilon_{12} > 0.025$.

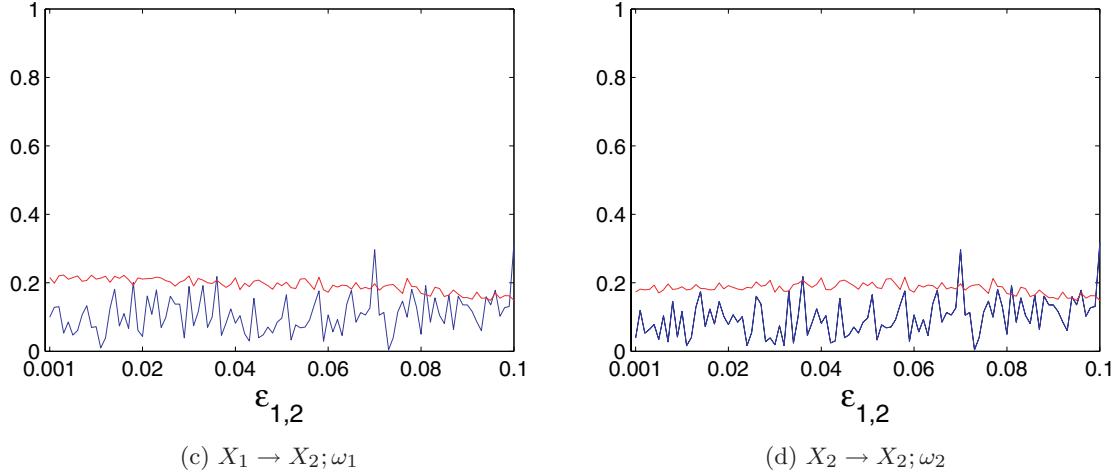


Fig. 3. (Continued)

corresponds to a weak coupling in the stochastic Rössler system (a, b). For the absent coupling direction, partial directed coherence is significant just for a few coupling strengths, which is expected for a 5%-significance level (c, d). For higher coupling strengths, corresponding to an almost complete synchronization between the oscillators, partial directed coherence will be significant for both directions.

4. Conclusion

Partial directed coherence is a powerful tool for the analysis of multivariate linear systems [Winterhalder *et al.*, 2005]. Moreover, we have shown that the linear concept of partial directed coherence can also be applied successfully to the nonlinear oscillatory system investigated here. For this, a high model order is required to model the systems by linear vector autoregressive processes sufficiently well, and consideration of the significance level is essential in order to prevent erroneous conclusions. By partial directed coherence analysis, not only a detection of coupling directions but also a differentiation of direct and indirect interactions is possible.

Acknowledgments

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