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PHASE SYNCHRONIZATION AND COHERENCE ANALYSIS: SENSITIVITY AND SPECIFICITY

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In Nonlinear Dynamics synchronization of oscillators is examined. Alternatively for linear stochastic systems, coherence analysis is utilized to detect interdependencies in transfer function systems. In contrast to the latter, oscillators continue oscillating in the absence of interaction between the processes. For transfer function systems the output ceases to exist without an input. Analysis techniques able to differentiate these considerably different classes of dynamics are desired in various applications. We show that conclusions from analysis techniques to the underlying dynamics have to be taken with care due to missing specificity. Moreover, we present an approach towards higher specificity.

Keywords: Nonlinear dynamics; sensitivity and specificity; coherence and phase synchronization.

1. Introduction

To investigate interrelations in multivariate systems, analysis techniques have been developed in various fields and are applied to empirical data, e.g. [Brockwell & Davis, 1998; Pikovsky *et al.*, 2001; Timmer *et al.*, 2000]. Detection of such interactions enables deeper insights into basic mechanisms and functioning underlying these systems. However, interpretations about the underlying dynamics of systems are based on solving an inverse problem which is usually more difficult than a direct problem. We show that for two widely used analysis techniques such conclusions about the underlying dynamics are impossible.

The nonparametric coherence analysis developed in the framework of linear stochastic systems enables a detection of interactions in transfer function systems [Brockwell & Davis, 1998]. A pointwise significance level prevents erroneous conclusions from finite time series generated by these processes. Coherence analysis has found wide-spread acceptability and has become a standard technique to analyze not only transfer function systems. Coherence analysis is very sensitive in detecting interactions in several systems but no conclusions about the underlying dynamics are possible from significant coherence values.

For nonlinear synchronizing oscillators, phase synchronization analysis has been shown to be very efficient in detecting interactions between oscillators [Pikovsky *et al.*, 2001]. Even for weak coupling and a possible presence of stochastic influences, phase synchronization analysis is able to reveal the coupling. This illustrates the high sensitivity of the method in analyzing interactions in multivariate systems. However, application of phase synchronization analysis to, for instance, a transfer function system will also show a significant result. Phase synchronization analysis is therefore not specific in detecting the correct class of underlying dynamics.

The noteworthy difference between the two classes of processes considered is the fact that two oscillators continue oscillating, independently when the coupling is absent. In contrast, the output of the transfer function system ceases to exist in case of no input. This difference may lead to remarkably different interpretations in applications to empirical data. Therefore, a methodology able to distinguish these two classes is desired. We demonstrate missing specificity for both analysis techniques by means of two genuine representative model systems of the two classes of dynamics, the transfer function systems and synchronizing oscillators. We propose an extension of both techniques to increase specificity without loss in sensitivity. By means of simulated data sets we show the performance of the proposed extension allowing the differentiation between the two types of dynamics.

2. Coherence and Phase Synchronization Analysis

Interactions in linear stochastic systems can be detected utilizing cross-spectral analysis. To this aim, the cross-spectrum $CS_{xy}(\omega) = \mathcal{FT}\{CCF_{xy}(\tau)\}$, which is the Fourier transform of the cross-covariance function of processes x and y, is normalized by the auto-spectra $S_{xx}(\omega) = \mathcal{FT}\{ACF_{xx}(\tau)\}$, which is the Fourier transform of the auto-covariance function of processes x and y, respectively, leading to the coherence function

$$\mathrm{Coh}_{xy}(\omega) = \frac{|CS_{xy}(\omega)|}{\sqrt{S_{xx}(\omega)\ S_{yy}(\omega)}} . \tag{1}$$

Coherence takes values of one in the case of a perfect linear interdependence between the processes x and y and values close to zero in absence of any interaction at frequency ω. When coherence has to be estimated for finite time series, a significance level prevents erroneous conclusions about the presence of interactions [Brockwell & Davis, 1998; Timmer *et al.*, 2000]. Coherence was estimated smoothing the auto- and cross-periodograms.

In order to detect phase synchronization between two coupled oscillatory systems, a suitable definition of phase and amplitude of a real-valued observed signal is necessary. To this aim, let $x(t)$ be the real-valued signal. The analytic signal is then given by $\psi(t) = x(t) + i\hat{x}(t) = A(t)\exp(i\varphi(t))$, where $A(t)$ is its amplitude and $\varphi(t)$ the phase [Gabor, 1946]. The imaginary counterpart of the analytic signal can be obtained by, e.g. the Hilbert transform of the signal [Oppenheim & Schafer, 1975]. Phase synchronization of two coupled, chaotic oscillators occurs if the $n : m$ phase locking condition $|n\varphi_x(t) - m\varphi_y(t)| = |\Phi_{n,m}| <$ const is satisfied [Pikovsky *et al.*, 2001] where $\varphi_x(t), \varphi_y(t)$ denote phases of the time series $x(t)$ and $y(t)$, for instance, the X-components of a Rössler oscillator, and n, m are given integers. To handle phase jumps, induced by the presence of dynamic or observation noise, $\Phi_{n,m}$ is modified by $\Psi_{n,m} = \Phi_{n,m} \bmod 2\pi$. In that case, a sharp peak in the distribution of $\Psi_{n,m}$ indicates phase synchronized oscillators [Tass *et al.*, 1998]. A commonly used quantity, measuring the sharpness of the distribution of $\Psi_{n,m}$, is the synchronization index [Mormann *et al.*, 2000; Pikovsky *et al.*, 2001]

$$R^2_{n,m} = \langle\cos(\Psi_{n,m})\rangle^2 + \langle\sin(\Psi_{n,m})\rangle^2 . \tag{2}$$

The synchronization index is $R_{n,m} = 1$ for a constant phase difference between the two oscillators and $R_{n,m} = 0$ for uniformly distributed phase differences.

3. Model Systems Under Investigation

In the following, the two model systems representing two classes of dynamics, i.e. a signal propagation paradigm and coupled self-sustained oscillators, are introduced. The first model system is a network of two coupled Rössler oscillators [Rössler *et al.*, 1976]

$$\begin{pmatrix} \dot{X}_j \\ \dot{Y}_j \\ \dot{Z}_j \end{pmatrix} = \begin{pmatrix} -\omega_j\, Y_j - Z_j + \left[\sum_i \varepsilon_{ji}\,(X_i - X_j)\right] + \sigma_j\, \eta_j \\ \omega_j\, X_j + a\, Y_j \\ b + (X_j - c)\, Z_j \end{pmatrix} \qquad i, j = 1, 2\,. \tag{3}$$

The parameters of the two oscillators are $\sigma_{1,2} = 1.5$, $\varepsilon_{12} = \varepsilon_{21} = 0.09$, $\omega_1 = 1.005$, $\omega_2 = 0.995$, $a = 0.15$, $b = 0.2$, $c = 10$ and $\eta_{1,2}$ denotes Gaussian distributed white noise. The Rössler system was sampled at 10 Hz.

Our exemplary transfer function system is a model for signal propagation based on a slightly modified autoregressive process of order one (AR[1])

$$u(t) = ru(t-1) + sx(t-\tau) + \eta(t)\,, \tag{4}$$

where the output signal $u(t)$ is a time-delayed and low-pass filtered version of the input signal $x(t)$ [Mertins, 1999]. The noise term $\eta(t)$ accounts for all influences which are not modeled by $x(t)$. Usually, $\eta(t)$ can be considered as Gaussian white noise accounting for various additional influences faced in real-world systems. Several choices for $x(t)$ are conceivable. We utilize the X-component of a single Rössler oscillator, sampled at 10 Hz, as input signal ($\sigma_1 = 1.5$, $\omega_1 = 1$, $a = 0.15$, $b = 0.2$, $c = 10$) and the parameters for the output signal are chosen to be $r = 0.4$, $s = 0.7$, and $\tau = 2$ in the following.

The difference between both model systems is the fact that the Rössler oscillators continue self-sustained oscillations regardless whether they are coupled or not. For the transfer function model, the output signal decreases exponentially in case of an absent input, i.e. there is no interrelation between input and output signals.

4. Sensitivity, Specificity, and Increased Specificity

To illustrate that both analysis techniques are sensitive in detecting the class of underlying dynamics, they are applied to the system they have been developed for. Coherence analysis is thus applied to the transfer function model and phase synchronization analysis to the coupled stochastic Rössler system. In Fig. 1 the results are shown for the signal propagation example and in Fig. 2 for the coupled stochastic Rössler system. Coherence is highly significant over a broad range of frequencies [cf. Fig. 1(a)]. Similarly, a sharp peak in the histogram of the phase differences strongly indicates phase synchronization [Fig. 2(a)]. Both techniques are thus able to reveal the interaction between the processes if the correct class of dynamics is present.

In Figs. 1(b) and 2(b) the results are shown for both analysis techniques when applied to the other model system. In the transfer function system, the histogram shows again a high concentration of phase differences strongly arguing for phase synchronization [Fig. 1(b)]. Thus, phase synchronization analysis is not specific in detecting the correct class of underlying dynamics as phase synchronization is based on coupled self-sustained oscillators. In the same way, coherence analysis is not specific in detecting transfer function systems. Coherence is highly significant over a broad range of frequencies when applied to the coupled stochastic Rössler system [cf. Fig. 2(b)].

In order to be able to draw conclusions about the underlying dynamics we suggest to apply coherence analysis to the phase fluctuations $\varphi_{1,2}(t) - \Omega_{1,2}t$, i.e. the phase of each signal subtracted by the linear trend caused by the frequency $\Omega_{1,2}$ obtained by the synchronized system. This is motivated by the fact that phase synchronization is a frequency related phenomenon. Coherence of the phase fluctuations for synchronizing oscillators is expected to be significant solely at the oscillation frequency if there is any significance at all. For propagated signals the phases of both, the input and the output, should be similar. This would lead to a broad band coherence between the phase fluctuations. Therefore, coherence analysis applied to the phase fluctuations should enable a distinction between both types of dynamics.

Moreover, in case of phase synchronizing oscillators and in presence of dynamic noise, phase jumps are expected to occur. Motivated by the washboard potential of the phase differences phase jumps in both directions are expected for sufficiently large noise variance. For signal propagation the output signal is expected to perform

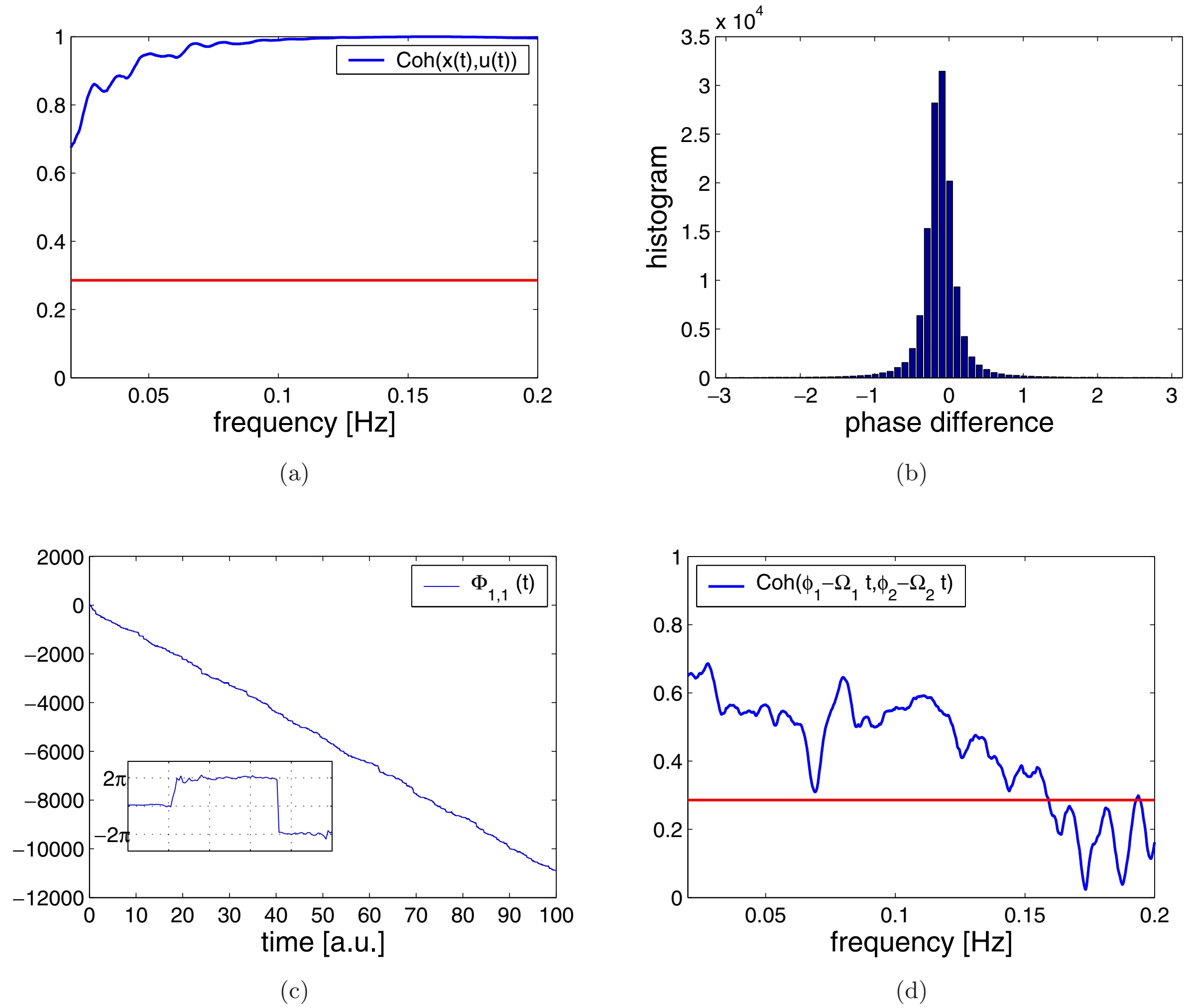

Fig. 1. Analysis of our transfer function model. (a) Coherence between the input signal $x(t)$ and the output signal $u(t)$. The 5% significance level is indicated by the red line and coherence is highly significant. (b) Histogram of the phase difference calculated between input signal $x(t)$ and the output signal $u(t)$. A sharp peak in the histogram is observed. (c) Time course of the phase difference and a corresponding magnification. Rapid phase changes in one direction are preferred for the transfer function model. (d) Coherence between the phase fluctuations (blue) and the 5% significance level (red). The coherence is significant in the low frequency range.

a phase jump after the input signal has performed one. Strictly speaking, there are no phase jumps for a single oscillator. Caused by trajectories close to the origin of the phase space, phase slips occur which are referred to as phase jumps in the following.

In Figs. 2(c) and 2(d), the results are shown for the coupled stochastic Rössler system. Phase jumps in both directions are observed with almost the same frequency (c) and coherence is significant solely at the oscillation frequency (d). In Figs. 1(c) and 1(d), the results are shown for our transfer function model. Phase jumps only in one direction occur (c) and coherence is significant over a broad range of frequencies (d). In summary, for a given high synchronization index, if coherence of the phase fluctuations is significant over a broad range of frequencies and if phase jumps appear merely in one direction, there is strong evidence for signal propagation. If in contrast coherence is only significant at the oscillation frequency and if phase jumps in both directions occur with almost the same frequency, there is strong evidence for synchronizing oscillators. For results that do not clearly fit into this scheme or for results obtained for low synchronization, no conclusion is feasible.

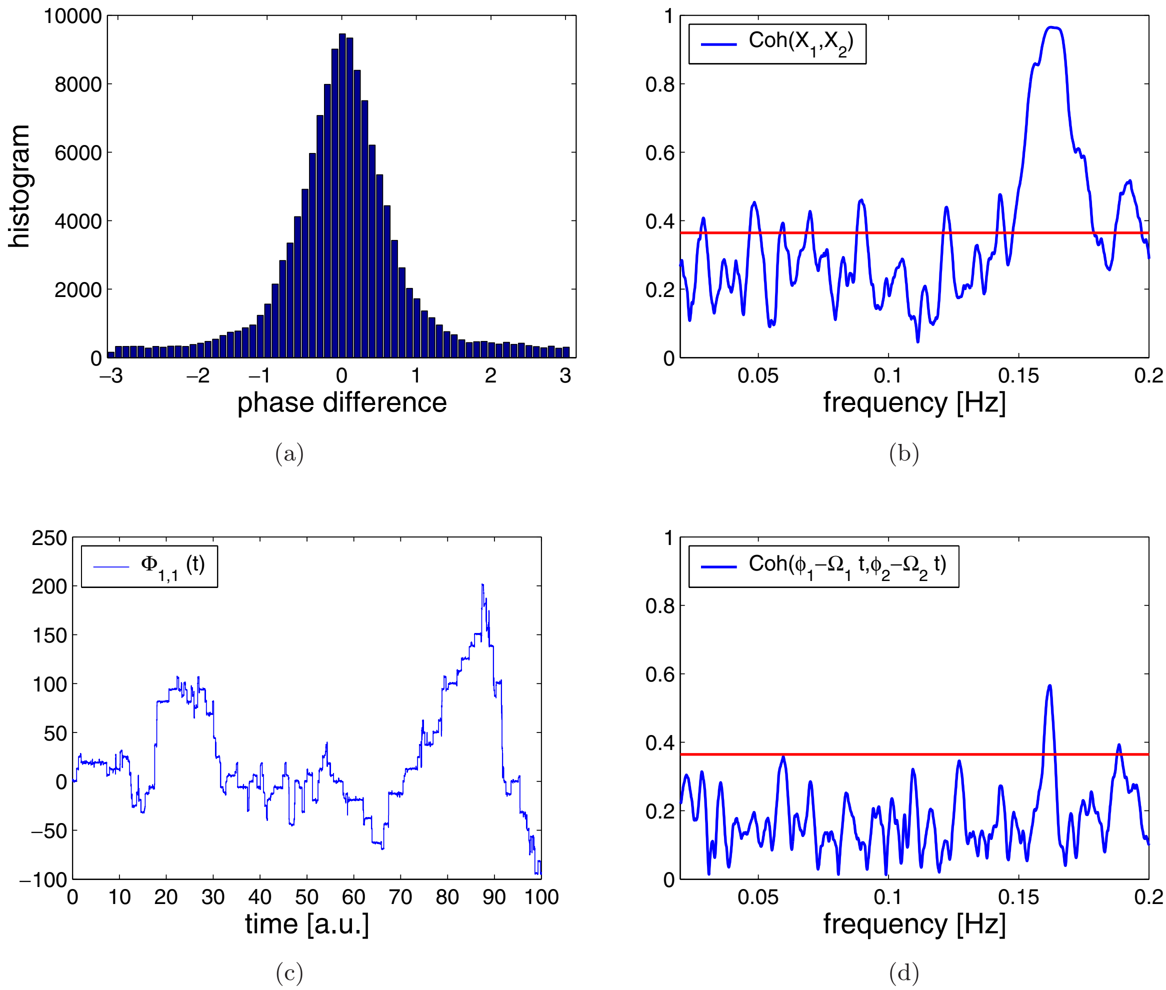

Fig. 2. Analysis of the coupled stochastic Rössler system. (a) Histogram of the phase difference calculated between the oscillators X_1 and X_2. A preferred value of the phase difference and therefore a phase synchronization between the oscillators is indicated. (b) Coherence between the oscillators X_1 and X_2. The 5% significance level is indicated by the red line and coherence is highly significant. (c) Time course of the phase difference. In this example, phase jumps in both directions occur with almost the same frequency. (d) Coherence between the phase fluctuations (blue) and the 5% significance level (red). The coherence is not significant in the low frequency range, only at the oscillation frequency.

5. Conclusion

By means of two illustrative and representative model systems, we have shown, that both coherence analysis as well as phase synchronization analysis are sensitive in detecting interactions in systems they have been developed for. In contrast, both analysis techniques do not prevent erroneous conclusions about the different dynamics when applied to the model systems they have not been developed for. In other words, if a conclusion about the underlying dynamics is the desired goal of the analysis, extensions of both techniques are required. To this aim, we proposed a combination of both concepts allowing differentiation of the two classes of dynamics.

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