Statistics and Numerics

Lecture Prof. Dr. Jens Timmer

Exercises Helge Hass, Mirjam Fehling-Kaschek

Exercise Sheet Nr. 6

Exercise 1: Bias and variance of ill-posed inverse problems

Create a function to compute the $N \times N$ Hilbert-matrix

$$A_{ij} = \frac{1}{i+j-1}$$

given the dimension N as input.

a) Generate simulated data $\vec{b}$ for normal distributed noise:

- choose:
$$x_j = \sin(2\pi(j-1)/(N-1)), \quad j = 1, \ldots, N$$
- compute:
$$\tilde{b}_i = \sum_{j=1}^{N} A_{ij} x_j \tag{1}$$
- add noise $\varepsilon_i \sim N(0, \sigma^2)$:
$$b_i = \tilde{b}_i + \varepsilon_i$$

b) Estimate $\vec{x}$ from the data $\vec{b}$:

- Calculate the singular-value decomposition $[U, s, V]$ of A (*numpy.linalg.svd*). How do the singular values s_i relate to the eigenvalues in case of a symmetric matrix?
- Compute the condition number $\kappa(A)$ via *numpy.amin* and *numpy.amax*.
- Write a function to estimate x_i for the given b_i from Eq. (1), using the inverted matrix A^{-1} based on its singular value decomposition:
$$A^{-1} = V^T [1/s_i] U^T \ . \tag{2}$$
Use the built-in functions *numpy.diag*, *numpy.divide*, *numpy.matmul* and *numpy.transpose*.

c) Test several setups regarding the effect of the regularization, $1/s_i = 0 \ if \ max(s)/s_i > \kappa_{max}$, for different choices of the maximal condition number κ_{max}, of N and of the noise σ.

- $N = 4$, $\sigma = 0.001$ for either no regularization or for $\kappa_{max} = 1000$
- $N = 7$, $\sigma = 10^{-5}$, $\kappa_{max} \in \{100, 10^5, 10^{10}\}$
- $N = 10$, $\sigma = 0$, $\kappa_{max} = inf$ (float("inf") in python). Compare the result with the one you obtain from using an in-build matrix inversion function (*numpy.linalg.inv*) instead of Eq. (2). How are A^{-1} and the x_i affected?
- $N = 42$, $\sigma = 0$, $\kappa_{max} \in \{100, 10^5, 10^{10}\}$

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